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DETERMINATION OF DIGITAL CONTROL SYSTEM RESPONSE BY CROSS-MULTI--ETC(U)
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DETERMINATION OF DIGITAL CONTROL SYSTEM RESPONSE BY CROSS-MULTIPLICATION

S.M. Seltzer
Guidance and Control Directorate

U.S. ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND

29 May 1979

Redstone Arsenal, Alabama 35809

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This report describes the Cross-Multiplication technique for obtaining the response of a digital control system from its closed-loop transfer function. When the latter is expressed in the complex z-domain, the response is obtained at the sampling instants. If it is desired to know the response between the sampling instants, the submultiple method may be adapted to the cross-multiplication technique. When the transfer function is expressed in the modified z-transform domain, the response is obtained at any point within each sampling instant. The three approaches are described and applied to an example.

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CONTENTS

1. INTRODUCTION

This report describes a technique for obtaining the response of a digital control system. It is assumed that the closed-loop transfer function is available in the z- or modified z-transform domain. The numerator and denominator of each side of the transfer function are cross-multiplied. The Real Translation Theorem is then applied to the result, yielding a difference equation in the time-domain. [1]This may be solved for the system response in terms of the reference (or other) input(s) to the system as well as in terms of system state initial conditions.

Two different modifications to the basic technique are described: one using the submultiple method and one using the modified z-transform technique. These are applied when it is desired to determine intra-sampling responses of the system. All three techniques are applied to a single example. A summary of the techniques and their application is provided at the conclusion of the report.

2. RESPONSE AT SAMPLING INSTANTS

It is assumed that a given digital or sampled-data system can be described by a closed-loop transfer function that relates the controlled output of the system to the reference input. If there is more than one input, the technique can also be applied to the resulting sum of closed-loop transfer functions relating the controlled output to each of the inputs. Although the report refers only to a single controlled output, the technique can be applied to find any system state if it is related to the inputs to the system in the z-domain. These relationships may be derived by using any of the standard techniques (such as signal flow graphs) or by the newly developed SAM (Systematic Analytical Method) technique[2].

It is assumed that the state whose response is desired is denoted in the z-domain as $C(z)$, where

$$C(z) \overset{d.}{=} \mathcal{Z}\left\{c(t)\right\} . \tag{1}$$

1. B.C. Kuo, *Analysis and Synthesis of Sampled-Data Control Systems*, Prentice-Hall, New Jersey, 1963.

2. S.M. Seltzer, *SAM: An Alternative to Sampled-Data Signal Flow Graphs*, US Army Missile Research and Development Command, Redstone Arsenal, Alabama, Technical Report T-79-49, May 1979.

The script z denotes the operation of taking the z-transform. Further assume that there is only one input into the system: R(z).

where

$$R(z) \overset{d}{=} \mathcal{Z}\left\{r(t)\right\} . \tag{2}$$

The relationship between C(z) and R(z) usually can be expressed as a closed-loop transfer function (or several such transfer functions) which is a ratio of two polynomials in z, i.e.,

$$\frac{C(z)}{R(z)} = \frac{\sum_{j=0}^{M} a_j z^j}{\sum_{k=0}^{N} b_k z^k} , \tag{3}$$

where coefficients a_j and b_j represent the system parameters. The procedure for finding the response at sampling instants — by the cross-multiplication method — consists of three steps:

Step 1. Cross-multiply the numerators with the denominators of Equation (3), yielding the expression,

$$\sum_{k=0}^{N} b_k z^k C(z) = \sum_{j=0}^{M} a_j z^j R(z) \tag{4a}$$

$$b_0 C(z) + b_1 z C(z) + \ldots + b_k z^k C(z) + \ldots + b_N z^N C(z)$$

$$= a_0 R(z) + a_1 z R(z) + \ldots + a_j z^j R(z) + \ldots + a_M z^M R(z) . \tag{4b}$$

Step 2. Each side of Equation (4a) or (4b) is divided by $b_N z^N$. The resulting equation is then solved for the C(z) term that is not multiplied by a non-zero power of z, i.e.,

$$C(z) = \frac{a_0}{b_N} z^{-N} R(z) + \frac{a_1}{b_N} z^{1-N} R(z) + \ldots + \frac{a_j}{b_N} z^{j-N} R(z) + \ldots$$

$$+ \frac{a_M}{b_N} z^{M-N} R(z) - \frac{b_0}{b_N} z^{-N} C(z) - \frac{b_1}{b_N} z^{1-N} C(z) - \ldots$$

$$- \frac{b_k}{b_N} z^{k-N} C(z) - \ldots - \frac{b_{N-1}}{b_N} z^{-1} C(z) \; . \tag{5}$$

Step 3. Apply the Real Translation Theorem to Equation (5), recalling that

$$\mathcal{Z}^{-1} \left\{ C(z) \right\} = c^*(t) \stackrel{d.}{=} \sum_{n=0}^{\infty} c(nT) \; \delta \; (t-nT) \tag{6a}$$

and

$$\mathcal{Z} \left\{ c(t-kT) \right\} = z^{-k} C(z) \, . \tag{6b}$$

The asterisk is used to indicate a variable that has been sampled, and $\delta(t-nT)$ denotes a Dirac function occuring at the instant $t=nT$. The resulting value of $c(nT)$ at each sampling instant, nT, then becomes

$$c(nT) = \frac{a_o}{b_N} r[(n-N)T] + \frac{a_1}{b_N} r[(n-N+1)T] + \ldots$$

$$+ \frac{a_j}{b_N} r[(n-N+j)T] + \ldots + \frac{a_M}{b_N} r[(n-N+M)T]$$

$$- \frac{b_o}{b_N} c[(n-N)T] - \frac{b_1}{b_N} c[(n-N+1)T] - \ldots$$

$$- \frac{b_k}{b_N} c[(n-N+k)T] - \ldots + \frac{b_{n-1}}{b_N} c[(n-1)T], \tag{7}$$

where j, k, M, N, and n are integers.

The advantages of the form of Equation (7) are three-fold:

- The value of c(nT), for any t=nT, may be obtained for any form of r(t), whether or not it is "z-transformable."
- The expression for c(nT) does not have to be recalculated every time the form of r(t) changes, as is the case when the response is determined by the partial fraction, power series, or inversion formula methods[1].
- The form of the expression for c(nT) permits the inclusion of initial conditions, such as c(0), if they exist.

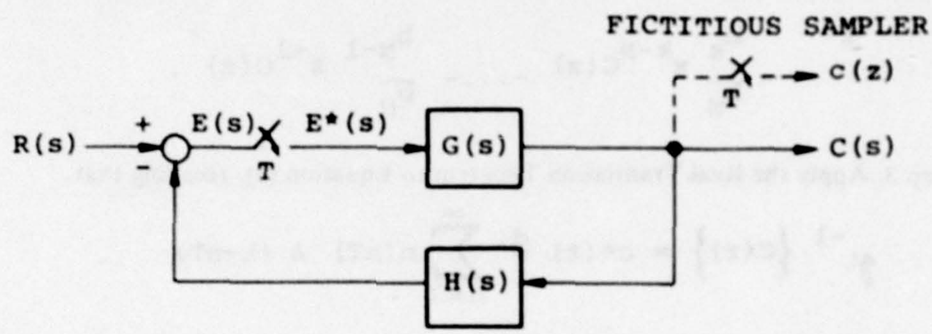

Figure 1. Closed-loop sampled-data system.

Example 1. Given the closed-loop sampled-data system of *Figure 1*, [1]the closed-loop transfer function easily is found to be:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + HG(z)} . \quad (8)$$

If G(s) and H(s) are given to be

$$G(s) = 1/\ s(s + 1)$$

and

$$H(s) = 1, \quad (9)$$

respectively, their z-transforms are

$$G(z) = \frac{z\,(1 - e^{-T})}{(z - 1)\,(z - e^{-T})} \,. \qquad (10)$$

and

$$HG(z) = G(z)\,. \qquad (11)$$

If one substitutes the expressions of Equations (*10*) and (*11*) into Equation (*8*), one obtains the closed-loop transfer function,

$$\frac{C(z)}{R(z)} = \frac{(1 - e^{-T})\,z}{z^2 - 2e^{-T}z + e^{-T}} \,. \qquad (12)$$

One may now apply the three steps prescribed for the cross-multiplication procedure, obtaining:

Step 1:

$$z^2C(z) - 2e^{-T}zC(z) + e^{-T}C(z) = (1-e^{-T})\;zR(z)\,. \qquad (13)$$

Step 2:

$$C(z) = (1 - e^{-T})z^{-1}R(z) + 2e^{-T}z^{-1}C(z) - e^{-T}z^{-2}C(z)\,. (14)$$

Step 3:

$$c(nT) = (1 - e^{-T})r\,[(n-1)T] + 2e^{-T}c\,[(n-1)T] - e^{-T}c\,[(n-2)T]\,. \qquad (15)$$

If, as in pp. 145-147 of Kuo's book, r(t) is assumed to be a unit step input, the system is assumed to be initially at rest, and the sampling period T is assumed to be 1 second, application of

Equation (15) readily yields the following values for $c(nT)$:

$c(T) = 0.6321$, the value of $c^*(t)$ at 1 sec.

$c(2T) = 1.0972$ the value of $c^*(t)$ at 2 sec.

$c(3T) = 1.2067$, the value of $c^*(t)$ at 3 sec.

$c(nT) = 0.6321 + 0.736\, c[(n-1)T] - 0.368[c\,(n-2)T]$, the value of $c^*(t)$ at $t = n$ sec.

These values correspond to those obtained by more tedious means in pages 145-147 of Kuo's book.

3. RESPONSE BETWEEN SAMPLING INSTANTS USING THE SUBMULTIPLE METHOD

If it is desired to find the intra-sampling response of the same type digital system described in Section 2, it may be accomplished by applying the submultiple method found in pages 83-86 of Kuo's book. Briefly, let $c\left[\frac{mT}{n}\right]$ represent the value of the response $c(t)$ at the instant, $t = \left[\frac{mT}{n}\right]$, where $n-1$ represents the number of intrasampling responses desired (n is an integer with value greater than unity). If m is an integer, the sampling period within which the submultiples are to be determined is denoted as mT. The z-transform of $c\left[\frac{mT}{n}\right]$ may be found from the ordinary z-transform in the following manner. In essence, $c\left[\frac{mT}{n}\right]$ is the output of a fictitious sampler which samples n times as fast as the real sampler. The z-transform of that output is defined as

$$\mathcal{Z}\left\{c\left(\frac{nT}{m}\right)\right\} \stackrel{d.}{=} C(z)_n = C(z)\Big|_{\substack{z\to z_n \\ T\to T_n}} \tag{16}$$

where

$$z_n = z^{1/n} \tag{17a}$$

and

$$T_n = T/n. \tag{17b}$$

Now the closed-loop expression of Equation (3) may be altered to read

$$\frac{C(z)_n}{R(z)} = \frac{\sum\limits_{j=0}^{M} a_j\, z_n{}^j}{\sum\limits_{k=0}^{N} b_k\, z_n{}^k}\,. \tag{18}$$

The submultiple modification to the basic method also consists of three steps.

Step 1. Cross-multiply the numerators and denominators of Equation (18), yielding the expression,

$$\sum_{k=0}^{N} b_k z_n^k C(z)_n = \sum_{j=0}^{M} a_j z_n^j R(z) \tag{19a}$$

or

$$b_0 C(z)_n + b_1 z_n C(z)_n + \ldots + b_k z_n^k C(z)_n + \ldots + b_N z_n^N C(z)_n$$

$$= a_0 R(z) + a_1 z_n R(z) + \ldots + a_j z_n^j R(z) + \ldots + a_M z_n^M R(z). \tag{19b}$$

Step 2. Each side of Equation (19a) or (19b) is divided by $b_N z^N$. The resulting equation is then solved for the $C(z)_n$ term that is not multiplied by a non-zero power of z_n, i.e.,

$$C(z)_n = \frac{a_0}{b_N} z_n^{-N} R(z) + \frac{a_1}{b_N} z_n^{-(N-1)} R(z) + \ldots$$

$$+ \frac{a_j}{b_N} z_n^{-(N-j)} R(z) + \ldots$$

$$+ \frac{a_M}{b_N} z_n^{-(N-M)} R(z) - \frac{b_0}{b_N} z_n^{-N} C(z)_n$$

$$- \frac{b_1}{b_N} z_n^{-(N-1)} C(z)_n - \ldots$$

$$- \frac{b_k}{b_N} z_n^{-(N-k)} C(z)_n - \ldots - \frac{b_{N-1}}{b_N} z_n^{-1} C(z)_n. \tag{20}$$

Step 3. Similar to the procedure of Section 2, the value of $c(t)$ at the n^{th} submultiple of the sampling instant mT is:

$$c\left(\frac{mT}{n}\right) = \frac{a_0}{b_N}\, r\left[\frac{(m-N)T}{n}\right] + \frac{a_1}{b_N}\, r\left[\frac{(m-N+1)T}{n}\right] + \ldots$$

$$+ \frac{a_j}{b_N}\, r\left[\frac{(m-N+j)T}{n}\right]$$

$$+ \ldots + \frac{a_n}{b_N}\, r\left[\frac{(m-N+M)T}{n}\right] - \frac{b_1}{b_N}\, c\left[\frac{(m-N+1)T}{n}\right] + \ldots$$

$$- \frac{b_k}{b_N}\, c\left[\frac{(m-N+k)T}{n}\right] - \ldots - \frac{b_{N-1}}{b_N}\, c\left[\frac{(m+1)T}{n}\right] . \qquad (21)$$

where all values of $r[.]$ are equal to zero except for those values of r at integral multiples of T, i.e.,

$$r(\ell T) = \begin{cases} 0 \;\forall\; \ell & \text{not integers} \\ r(t)\Big|_{t=\ell T} & \text{for } \ell = 0, 1, 2, \ldots \text{(integers)} \end{cases} \qquad (22)$$

and j, k, M, m, N, n are integers.

Example 2. The same sampled-data system is used in this example as in Example 1. It may be shown, for the example at hand, that

$$C(z)_n = G(z)_n\; E(z), \qquad (23)$$

where $G(z)_n$ is found in the same manner as demonstrated in Equation (16). From *Figure 1* it is seen that

$$E(z) = \frac{R(z)}{1 + HG(z)} . \qquad (24)$$

Substitution of Equation (24) into Equation (23) leads to

$$\frac{C(z)_n}{R(z)} = \frac{G(z)_n}{1 + HG(z)} . \qquad (25)$$

Replacing z by z_n and T by T_n in Equation (10) leads to the expression,

$$G(z)_n = \frac{z_n(1 - e^{-T_n})}{(z_n - 1)\ (z_n - e^{-T_n})} \quad . \tag{26}$$

If one substitutes Equations (17) into Equation (11) and substitutes the resulting equation and Equation (26) into Equation (25), one obtains the closed-loop expression,

$$\frac{C(z)_n}{R(z)} = \frac{z_n(1 - e^{-T_n})\ (z - 1)\ (z - e^{-T})}{(z_n-1)(z_n-e^{-T_n})(z_n^{2n}-2e^{-T}z_n^{n}+e^{-T})} \quad . \tag{27}$$

The denominator of this expression is of course (when equated to zero) the characteristic equation. It may be expanded into the following polynomial in z_n:

$$\begin{aligned} D(z_n) = {} & z_n^{2n+2} - (1 + e^{-T_n})\ z_n^{2n+1} + e^{-T_n}\ z_n^{2n} \\ & - 2e^{-T}z_n^{n+2} + 2e^{-T}\ (1 + e^{-T_n})\ z_n^{n+1} - 2e^{-(T+T_n)}z_n^{n} \\ & + e^{-T}z_n^{2} - e^{-T}\ (1 + e^{-T_n})\ z_n + e^{-(T+T_n)} \quad . \end{aligned} \tag{28}$$

Similarly, the numerator of Equation (27) may be expressed as the polynomial,

$$N(z_n) = (1 - e^{-T_n})\left[z_n^{2n+1} - (1 + e^{-T})\,z_n^{n+1} + e^{-T}z_n\right] \quad . \tag{29}$$

Step 1. If Equations (28) and (29) are substituted into Equation (27) and the resulting numerators and denominators cross-multiplied, one obtains the equivalent of Equation (19b):

$$
\begin{aligned}
&z_n^{2n+2}C(z)_n - (1 + e^{-T_n})\, z_n^{2n+1}C(z)_n + e^{-T_n} z_n^{2n}C(z)_n \\
&- 2e^{-T} z_n^{n+2}C(z)_n + 2e^{-T}(1 + e^{-T_n}) z_n^{n+1}C(z)_n \\
&-2e^{-(T+T_n)} z_n^{n}C(z)_n \\
&+ e^{-T} z_n^{2}C(z)_n - e^{-T}(1 + e^{-T_n}) z_n C(z)_n + e^{-(T+T_n)}C(z)_n \\
&= (1 - e^{-T_n})\, [z_n^{2n+1}R(z) - (1 + e^{-T}) z_n^{n+1}R(z) \\
&+ e^{-T} z_n\, R(z)]\,.
\end{aligned}
\tag{30}
$$

Step 2. If one applies the procedure of Step 2 to Equation (30) one obtains

$$
\begin{aligned}
C(z)_n &= (1 - e^{-T_n})\, [z_n^{-1}\, R(z) - (1 + e^{-T}) z_n^{-(n+1)}R(z) \\
&\quad + e^{-T} z_n^{-(2n+1)}\, R(z)] + (1 + e^{-T_n}) z_n^{-1}C(z)_n \\
&\quad - e^{-T_n} z_n^{-2}C(z)_n + 2e^{-T} z_n^{-n}C(z)_n \\
&\quad - 2e^{-T}(1+e^{-T_n}) z_n^{-(n+1)}C(z)_n + 2e^{-(T+T_n)} z_n^{-(n+2)}C(z)_n \\
&\quad - e^{-T} z_n^{-2n}C(z)_n + e^{-T}(1 + e^{-T_n}) z_n^{-(2n+1)}C(z)_n \\
&\quad - e^{-(T+T_n)} z_n^{-(2N+2)}C(z)_n\,.
\end{aligned}
\tag{31}
$$

Step 3. Finally, one may apply the procedure of Step 3 to obtain the time-domain difference equation that yields the value of $c^*(t)$ at the instant of time, $t = \frac{mT}{n}$, where m is any desired integer.

$$c\left(\frac{mT}{n}\right) = (1 - e^{-Tn}) \left\{ r\left[\frac{(m-1)T}{n}\right] - (1 + e^{-T}) r\left[\frac{(m-n+1)T}{n}\right] \right.$$

$$\left. + e^{-T} r\left[\frac{(m-2n-1)T}{n}\right] \right\}$$

$$+ (1 + e^{-Tn}) c\left[\frac{(m-1)T}{n}\right] - e^{-Tn} c\left[\frac{(m-2)}{n}T\right]$$

$$+ 2e^{-T} c\left[\frac{(m-n)T}{n}\right]$$

$$- 2e^{-T}(1+e^{-Tn}) c\left[\frac{(m-n-1)T}{n}\right] + 2e^{-(T+Tn)} c\left[\frac{(m-n-2)T}{n}\right]$$

$$- e^{-T} c\left[\frac{(m-2n)T}{n}\right] + e^{-T}(1 + e^{-Tn}) c\left[\frac{(m-2N-1)T}{n}\right]$$

$$- e^{-(T+T_n)} c\left[\frac{(m-2n-2)T}{n}\right]. \qquad (32)$$

Again, as in Kuo's book and in Example 1, it is assumed the system is at rest initially, and the sampling period T is one second. One may desire to know two intra-sampling values of the output $c^*(t)$. In that case, $n = 2 + 1 = 3$. If one substitutes these numerical values into Equation (32), one obtains the equation,

$$c\left(\frac{m}{3}\right) = 0.2835 r\left(\frac{m-1}{3}\right) - 0.4866\ r\left(\frac{m-4}{3}\right)$$

$$+ 0.1043 r\left(\frac{m-7}{3}\right) + 1.7165\ c\left(\frac{m-1}{3}\right)$$

$$-0.7165 c\left(\frac{m-2}{3}\right) + 0.5272 c\left(\frac{m-3}{3}\right) - 1.2630 c\left(\frac{m-4}{3}\right)$$

$$+0.5272 c\left(\frac{m-5}{3}\right)$$

$$-0.3679 c\left(\frac{m-6}{3}\right) + 0.6315 c\left(\frac{m-7}{3}\right) - 0.2636 c\left(\frac{m-8}{3}\right). \qquad (33)$$

It is worthwhile to pause a moment and consider the power of the difference equation just obtained. First, the output c $\left[\frac{m}{3}\right]$ can be obtained for any deterministic reference input r(t). It only need be specified at the sampling instants. Second, initial conditions or instantaneous changes can be accommodated readily by the equation. Third, difference equations are particularly amenable to programming on desktop calculators or digital computers. These three advantages are not enjoyed by all methods found in the standard textbooks for determining the state(s) of a sampled-data system. The submultiple method does suffer from the drawback of having to specify beforehand the number n-1 of the intrasampling instants for which it is desired to know a given state, such as c(t) in this case.

In the case at hand, a brief demonstration of the calculation of c(t) at instants t=0, T/3, 2T/3, T,..., is provided below. A unit step input is assumed so that the results may be compared to those of Kuo in pages 195-198 of his book. The symbol c $\left[\frac{m}{3}\right]$ denotes the value of c(t) at t= $\left[\frac{m}{3}\right]$ sec.

m=0: c(0) = 0
m=1: c(1/3) = 0.2835 r(0) + 1.7165 c(0) = 0.2835
m=2: c(2/3) = 0.2835 r(1/3) + 1.7165 c(1/3) = 0.4866
m=3: c(1) = 0.2835 r(2/3) + 1.7165 c(2/3) - 0.7165 c(1/3) = 0.6321
m=4: c(4/3) = 0.2835 r(1) - 0.4866 r(0) + 1.7165 c(1) - 0.7165 c(2/3) + 0.5272 c(1/3) = 0.8407
m=5: c(5/3) = 0.2835 r(4/3) - 0.4866 r(1/3) + 1.7165 c(4/3) - 0.7165 c(1) + 0.5272 c(2/3) - 1.2630 c(1/3) = 0.9901
m=6: c(2) = 1.0972
m=7: c(7/3) = 1.1464
m=8: c(8/3) = 1.1816
m=9: c(3) = 1.2067

4. RESPONSE BETWEEN SAMPLING INSTANTS USING MODIFIED Z-TRANSFORM METHOD

If it is desired to find the intra-sampling response of the same type of digital system described in Section 2, it also may be accomplished by applying the modified *z*-transform method. In this method it is necessary first to determine the modified *z*-transform equivalent of the closed-loop transfer function of Equation (3). The reader's mind will be refreshed (as needed).

Let c(t) denote the response of a digital system and $c^*(t)$ its sampled output. The value of c(t) at the instant of time, $t=(n-\Delta)T$, is the value of c(t) that has been delayed by an increment of time ΔT. The latter is represented symbolically as $c(t-\Delta T)$. If the relation,

$$m = 1 - \Delta, \qquad 0 \le m \le 1 , \tag{34}$$

is used, the value of c(t) delayed by an amount ΔT after the sampling instant, $t=nT$, may be denoted as $c[(n-1+m)T]$. The z-transform of $c[(n-1+m)T]$ is termed the modified z-transform ($\mathcal{Z}_m$) of c(t), i.e.,

$$\mathcal{Z}\{c(t-T)\} = \mathcal{Z}\{c[(n-1+m)T]\} =$$

$$\mathcal{Z}_m\{c(t)\} \overset{d.}{=} C(z,m) = C(z,\Delta)\Big|_{\Delta = 1-m} . \tag{35}$$

It also may be shown that

$$C(z,m) = z^{-1}\,\mathcal{Z}\{c(t+mT)\} = z^{-1}\sum_{k=0}^{\infty} c\ [(m+n)T]\ z^{-n}. \tag{36}$$

Alternately, C(z,m) may be derived through its complex convolution definition:

$$C(z,m) = \mathcal{L}\{c(t-\Delta T)\times\delta_T(t)\}\Big|_{z=e^{Ts}}$$

$$= \Big[\mathcal{L}\{c(t-\Delta T)\}\star\mathcal{L}\{\delta_T(t)\}\Big]\Big|_{z=e^{Ts}} \tag{37}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} C(\xi)C^{-\Delta T\xi}\left[1-e^{-T(s-\xi)}\right]^{-1}d\xi\Big|_{z=e^{Ts}},$$

where $\delta_T(t)$ represents a train of impulses each with a unit area, the star symbol (★) denotes the complex convolution operation, $\mathcal{L}$ denotes the operation of taking the Laplace transform, c denotes the abscissa of convergence (a positive real number), and ξ represents a dummy complex variable.

Equations (35) - (37) represent two methods of determining analytically the modified z-transform (it is recommended that one look it up in a modified z-transform table, if one is

available). The above relationships may also be used to determine the modified z-transform of the transfer function, $G(z, m)$.

For the purpose of exposition, assume the modified z-transform equivalent of the closed-loop transfer function is known and is

$$\frac{C(z,m)}{R(z)} = \frac{\sum_{j=0}^{M} a_j \; z^j}{\sum_{k=0}^{N} b_k \; z^k} , \tag{38}$$

where a_i and b_i may be functions of m. The cross-multiplication technique is similar to those two previously described.

Step 1. Cross-multiply the numerators and denominators of Equation (38), yielding the expression.

$$\sum_{k=0}^{N} b_k \; z^k \; C(z,m) = \sum_{j=0}^{M} a_j z^j R(z) \tag{39a}$$

$$b_0 C(z,m) + b_1 z C(z,m) + b_2 z^2 C(z,m) + \ldots + b_k z^k C(z,m)$$

$$+ \ldots + b_N z^N C(z,m) = a_0 R(z) + a_1 z R(z) + \ldots + a_j z^j R(z)$$

$$+ \ldots + a_M z^M R(z) . \tag{39b}$$

Step 2. Each side of Equation (39a) or (39b) is divided by $b_N z^N$. The resulting equation is then solved for the $C(z, m)$ term that is not multiplied by a non-zero of z, i.e.,

$$C(z,m) = \frac{a_0}{b_N} z^{-N} R(z) + \frac{a_1}{b_N} z^{-(N-1)} R(z) + \ldots + \frac{a_j}{b_N} z^{-(N-j)} R(z)$$

$$+ \ldots + \frac{a_M}{b_N} z^{-(N-M)} R(z) - \Bigg[\frac{b_0}{b_N} z^{-N} C(z,m)$$

$$+ \frac{b_1}{b_N} z^{-(N-1)} C(z,m) + \ldots + \frac{b_k}{b_N} z^{-(N-k)} C(z,m) + \ldots$$

$$\left. + \frac{b_{N-1}}{b_N} z^{-1} C(z,m) \right] . \tag{40}$$

Step 3. The inverse z-transform of each term of Equation (40) is determined, using the inverse of Equation (39b).

$$\begin{aligned} c\left[(n-1)T,m\right] \equiv c\left[(n-1+m)T\right] &= \frac{a_0}{b_N} r\left[(n-N)T\right] + \frac{a_1}{b_N} r\left[(n-N+1)T\right] + \\ &\ldots + \frac{a_j}{b_N} r\left[(n-N+j)T\right] + \ldots + \frac{a_N}{b_N} r\left[(n-N+m)T\right] \\ &- \left\{ \frac{b_0}{b_N} c\left[(n-1+m-N)T\right] + \frac{b_1}{b_N} c\left[(n+m-N)T\right] + \ldots \right. \\ &\left. + \frac{b_k}{b_N} c\left[(n-1+m-N+k)T\right] + \ldots + \frac{b_{N-1}}{b_N} c\left[(n-2+m)T\right] \right\} \end{aligned} \tag{41}$$

where j, K, m, M, N are integers.

Example 3. Again, the same sampled-data system is used in this example as in Examples 1 and 2. The modified z-transform method will be applied. A fictitious time delay, $e^{-(1-m)}$, is placed following the forward loop gain, G(s), and a fictitious time advance (of the same magnitude as the time delay) is placed just before the feedback loop gain H(s). The closed-loop transfer function that results from these two addition elements being added is

$$\frac{C(z,m)}{R(z)} = \frac{G(z,m)}{1 + HG(z)} . \tag{42}$$

Using Equation (9) for G(s), one may obtain G(z,m):

$$\begin{aligned} G(z,m) &= \mathcal{Z}_m\left\{G(s)\right\} = \mathcal{Z}_m\left\{\frac{1}{s(s+1)}\right\} \\ &= \frac{(1-e^{-mT})z + (e^{-mT}-e^{-T})}{(z-1)(z-e^{-T})} . \end{aligned} \tag{43}$$

The value for HG(z) is given in Equation (9) - (10) which, with Equation (43), may be substituted into Equation (42) to yield

$$\frac{C(z,m)}{R(z)} = \frac{(1-e^{-mT})z + (e^{-mT}-e^{-T})}{(z-1)(z-e^{-T})\left[1 + \frac{(1-e^{-T})z}{(z-1)(z-e^{-T})}\right]}$$

$$= \frac{(1-e^{-mT})z + (e^{-mT} - e^{-T})}{z^2 - 2e^{-T}z + e^{-T}} \ . \tag{44}$$

It may be observed that Equation (44) is in the same form as Equation (38). Hence the operations denoted as Steps 1, 2, and 3 (above) may be applied.

Step 1. The numerators and denominators of Equation (44) are cross-multiplied, yielding

$$z^2\ C(z,m) - 2e^{-T}\ z\ C(z,m) + e^{-T}\ C(z,m)$$

$$= (1-e^{-mT})z\ R(z) + (e^{-mT} - e^{-T})z^{-2}R(z) \ . \tag{45}$$

Step 2. Each term of Equation (45) is divided by z^2, and the resulting equation is rearranged to solve for C(z, m):

$$C(z,m) = (1-e^{-mT})z^{-1}R(z) + (e^{-mT}-e^{-T})z^{-2}R(z)$$

$$+ 2e^{-T}z^{-1}C(z,m) - e^{-T}z^{-2}C(z,m) \ . \tag{46}$$

Step 3. Equation (46) is now transformed into the time domain, resulting in the following difference equation:

$$c[(n-1+m)T] = (1-e^{-mT})\ r[(n-1)T] + (e^{-mT}-e^{-T})\ r[(n-2)T]$$

$$+ 2e^{-T}\ c[(n-2+m)T] - e^{-T}\ c[(n-3+m)T] \ . \tag{47}$$

During the n^{th} sampling period, i.e., $(n-1)T \leq t \leq nT$, where n is an integer, one lets m assume values between zero and one to find the intrasampling values of c(t). For instance if it is desired to verify the values obtained in Example 2, let m=1/3 and 2/3 and vary the integer n incrementally, starting at n=0. Again, for simplicity, let T = 1 second

$$n = 0: \quad c(m-1) = 0$$

$$n = 1: \quad c(m) = (1-e^{-m})\, r(0) = 1-e^{-m}$$

$$m = 0 \text{ (check case)}: \quad c(0) = 0$$

$$m = 1/3: c(1/3) = 1 - 0.7165 = 0.2835$$

$$m = 2/3: c(2/3) = 0.4866$$

$$m = 1: c(1) = 0.6321$$

$$n = 2: \quad c(1+m) = (1-e^{-m})\, r(1) + (e^{-m}-e^{-1})\, r(0) + 2e^{-1}\, c(m)$$

$$m = 0 \text{ (check case)}: c(1) = 0.6321$$

$$m = 1/3: c(4/3) = (1-e^{-1/3}) + (e^{-1/3}-e^{-1}) + 2e^{-1}\, c(1/3) = 0.8407$$

$$m = 2/3: c(5/3) = (1-e^{-2/3}) + (e^{-2/3}-e^{-1}) + 2e^{-1}\, c(2/3) = 0.9901$$

$$m = 1: c(2) = (1-e^{-1}) + (e^{-1}-e^{-1}) + 2e^{-1} = 1.0972$$

$$n = 3: c(2+m) = (1-e^{-m}) r(2) + (e^{-m}-e^{-1}) r(1)$$

$$+ 2e^{-1} c(1+m) - e^{-1} c(m)$$

$$m = 0 \text{(check case)}: c(2) = (1-e^{-0})$$

$$+ (e^{-0}-e^{-1})$$

$$+ 2e^{-1} c(1) - e^{-1} c(0)$$

$$= 1.0972$$

$$m = 1/3: c(7/3) = 1-e^{-1/3} + e^{-1/3}-e^{-1}$$

$$+ 2e^{-1} c(4/3) - e^{-1} c(1/3)$$

$$= 1.1464$$

$$m = 2/3: c(8/3) = 1-e^{-2/3} + e^{-2/3} - e^{-1}$$

$$+ 2e^{-1} c(5/3) - e^{-1} c(2/3)$$

$$= 1.1816$$

$$m = 1: c(3) = 1-e^{-1} + e^{-1} - e^{-1} + 2e^{-1} c(2)$$

$$- e^{-1} c(1)$$

$$= 1.2067$$

etc.

5. CONCLUSIONS

Several analytical techniques for obtaining the response of a digital control system have been described. They are based on a single principle: cross-multiplication followed by applications of the real translation theorems. Each is applied to a single example. As a starting point for application of each of the techniques, it is required that the dynamics of the digital control system be described in the z- or modified z-domain.

The advantages of the three techniques over extant classical methods are:

- The response may be obtained for any deterministic reference input into the system as long as its value is known at the sampling instants. It need not be described by a differential equation, and the z-transform for a specific reference input need not be determined before obtaining an expression for the response.
- Initial conditions and instantaneous changes can be accommodated readily by the equations obtained through use of the cross-multiplication methods.
- The difference equations obtained are particularly amenable to programming on a desktop calculator or digital computer.
- A detailed knowledge of the theory underlying digital or sampled-data control systems is not required (although it certainly is helpful) by the analyst in order to apply the recipes described herein.

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